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The motion of charged particles in a rotating plasma

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With 1 figure in the text

ABSTRACT

A reconsideration is made of the confinement of charged particles by magnetic mirrors in a rotating plasma. The forbidden regions of a charged particle in crossed electric and magnetic fields are discussed with special attention to the centrifugal and Coriolis forces. It is found that, under certain conditions, particles will move almost along the magnetic field lines and will be completely reflected at the mirrors by means of the centrifugal force. A high reflection power is obtained by fields with large radial extensions such as that generated by a circular current loop.

1. Introduction

Thermonclear devices based upon the plasma drift in crossed electric and magnetic fields have recently been discussed by a number of investigators. In the "Homopolar" machine conceived by Baker and Anderson in 1956 (Anderson *et al.* 1958) as well as in the "Ixion" by Boyer *et al.* (1958) the fields applied are used for both containment and heating of the plasma. In an ordinary mirror machine a considerable fraction of particles will be able to escape through the mirrors. However, Baker and Anderson have pointed out that the rotation of a plasma by means of crossed fields produces a centrifugal force and a "gravitational" effect which has a tendency to keep particles of both signs from approaching the machine axis at the mirrors. In the limit of weak mirror ratios it was found that an enhanced, but not complete, containment could be achieved by means of the rotation. A mathematical analysis was performed by Boyer *et al.* (1958) leading to the same result. In the analysis it was assumed that the angular velocity of rotation is much smaller than the gyro frequency of ions and that the electric and magnetic fields are proportional to $1/r$ and $1/r^2$, where r is the distance from the axis of the machine.

A magnetic bottle produced by the field from a current loop can be used as a mirror configuration with very high mirror ratios. This is done by extending the leads of the loop radially inwards through the strongest part of the field as shown in Fig. 1. It has been suggested by one of the authors of the present paper (B. B.) to use the centrifugal force in a rotating plasma to keep particles away from the lead region.

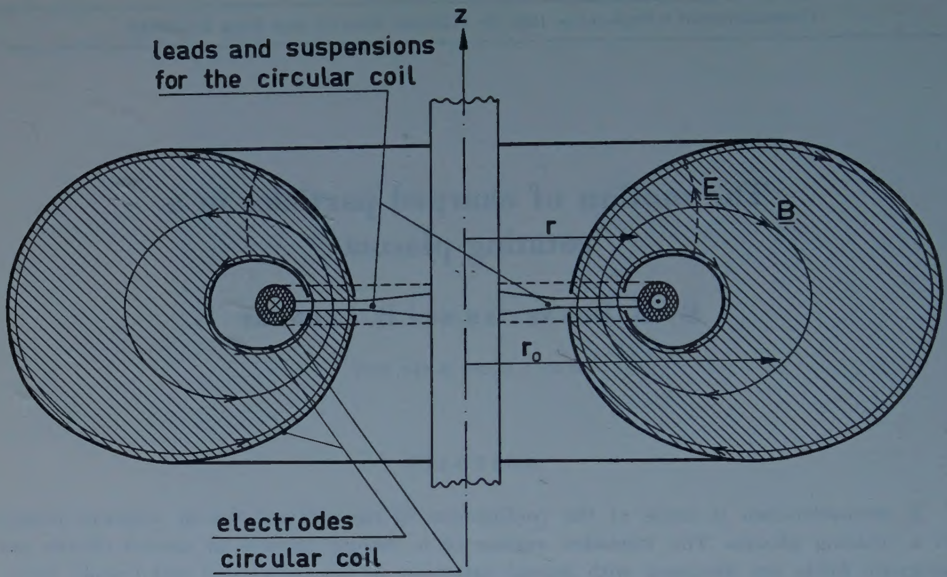

Fig. 1. Charged particles are trapped in the shaded area by the magnetic field ***B*** produced by a circular coil. An electric field ***E*** is introduced which produces a rotation arond the axis of symmetry and prevents particles from approaching the leads of the coil.

The following considerations will show that, at least under certain conditions, a rotation around the axis of symmetry will not only reduce the mirror losses to the leads but will make the confinement complete. This is also imaginable from the macroscopic point of view. Fig. 1 may be considered as a "bag" of magnetic field lines within which plasma is rotated around the axis. If the bag is strong enough motion of matter inside it will only be distorted little by Coriolis force and matter will almost be guided along the magnetic field lines. However, matter starting at the "bottom" of the bag is situated in a gravitation field produced by the centrifugal force. It can be lost only by being "lifted" all the way from the bottom up to the leads, just like a particle which escapes from a planet by performing all the work against its gravitational field. This will be possible only if the thermal velocity of the particle exceeds the escape velocity, the latter being determined by the strength of the gravitational field.

2. Motion of a particle in a rotating system

In the present paper a study will be undertaken of the motion of a particle with charge q and mass m in a stationary electric field

$$\boldsymbol{E}=\left(-\frac{\partial \phi}{\partial r},\ 0,\ -\frac{\partial \phi}{\partial z}\right) \tag{1}$$

and a stationary poloidal magnetic field

$$\boldsymbol{B} = \text{curl } \boldsymbol{A} = \left(-\frac{\partial A_\varphi}{\partial z},\ 0,\ \frac{1}{\mathrm{r}} \cdot \frac{\partial}{\partial r} (r A_\varphi) \right), \tag{2}$$

where cylindrical coordinates have been introduced and both fields are assumed to be axially symmetric. The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are measured in a coordinate system at rest with respcet to the observer (laboratory system). In this system the particle is observed to move with the absolute velocity $\boldsymbol{u}$, which is much less than the velocity of light. Further, introduce a coordinate system with the same origin and the same z-axis as the laboratory system and which rotates around the z-axis with the angular velocity $\boldsymbol{\Omega}$. Then,

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{R}, \tag{3}$$

where $\boldsymbol{v}$ is the relative velocity of the particle measured in the rotating system and $\boldsymbol{R}$ is the vector from the origin in the laboratory system which indicates the position of the particle. If the corresponding position vector in the rotating system is defined by $\boldsymbol{r}$ the absolute acceleration of the particle can be written as

$$\frac{d\boldsymbol{u}}{dt} = \frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}, \tag{4}$$

where the Coriolis and centrifugal accelerations are given by the second and third terms of the right hand member. The electric field measured in a coordinate system which follows the particle becomes

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \tag{5}$$

and the equation of motion

$$m \frac{d\boldsymbol{u}}{dt} = q\boldsymbol{E}'. \tag{6}$$

To any point in the rz-plane is now associated a local coordinate system rotating with an angular velocity $\boldsymbol{\Omega} = \boldsymbol{\Omega}(r, z)$. A special example is given by a homopolar device, where

$$\boldsymbol{\Omega} \times \boldsymbol{R} = (\boldsymbol{E} \times \boldsymbol{B})/B^2. \tag{7}$$

This specifies Ω as a function of the given fields, i.e., by $\boldsymbol{E}$ and $\boldsymbol{B}$. From eqs. (5) and (7)

$$\boldsymbol{E}' = \boldsymbol{E}_{\parallel} + \boldsymbol{v} \times \boldsymbol{B}, \tag{8}$$

where $\boldsymbol{E}_{\parallel}$ is the electric field component along the magnetic field.

In the φ-direction the equation of motion (6) can be combined with eqs. (2) and (4) giving the result

$$\frac{d}{dt}\left(r v_\varphi + \frac{q}{m} r A_\varphi + \Omega r^2 \right) = 0. \tag{9}$$

With values at the starting point indicated by index $(_0)$ eq. (9) is integrated to

$$r v_\varphi = r_0 v_{\varphi_0} + \frac{q}{m}(r_0 A_{\varphi_0} - r A_\varphi) + \Omega_0 r_0^2 - \Omega r^2. \tag{10}$$

This result is valid even without the restriction given by eq. (7).

Scalar multiplication of the equation of motion (6) by $\boldsymbol{u}$ gives

$$\frac{d}{dt}(\tfrac{1}{2} m \boldsymbol{u}^2) - q \boldsymbol{E} \cdot \boldsymbol{u} = 0 \tag{11}$$

and

$$u^2 = u_0^2 + \frac{2q}{m}(\phi_0 - \phi). \tag{12}$$

Eq. (10) expresses conservation of momentum around the axis and eq. (12) conservation of energy. These relations can also be deduced from the Hamiltonian formalism. Finally, combination of expressions (10), (3) and (12) results in

$$v_r^2 + v_z^2 = v_0^2 + 2\Omega_0 r_0 v_{\varphi_0} + \Omega_0^2 r_0^2 + \frac{2q}{m}(\phi_0 - \phi) - \frac{1}{r^2}\left[r_0 v_{\varphi_0} + \frac{q}{m}(r_0 A_{\varphi_0} - r A_\varphi) + \Omega_0 r_0^2\right]^2. \tag{13}$$

A relation corresponding to eq. (13) but expressed in terms of the total velocity $\boldsymbol{u}$ has earlier been found to have the form (Lehnert 1959)

$$u_r^2 + u_z^2 = u_0^2 + \frac{2q}{m}(\phi_0 - \phi) - \frac{1}{r^2}\left[r_0 u_{\varphi_0} + \frac{q}{m}(r_0 A_{\varphi_0} - r A_\varphi)\right]^2. \tag{14}$$

The expression (13) can also be derived from eqs. (14) and (3). The form (13) where the rotation has been separated from the excess velocity $\boldsymbol{v}$, makes it possible to derive conclusions which are not obvious from a pure inspection of eq. (14), even if they are intrinsic in the underlying theory.

3. Forbidden regions

Since the left hand sides of eqs. (13) and (14) cannot be negative an examination of the sign of their right hand sides can be used to determine the forbidden regions of the particle motion. Areas where the right hand sides are negative cannot be reached by the particle. However, it should be stressed that the results (13) and (14) do not specify the particle path and contain less information than the equation of motion. Thus, a positive value of the right hand sides at a point (r, z) does not necessarily imply that the particle is allowed to reach (r, z) from (r_0, z_0). Due to the separation of rotation and excess velocity, however, eq. (13) contains more information and is more restrictive than eq. (14).

If the somewhat artificial situation of rigid body rotation is used as an illustration to eq. (13) the result can be written as

$$v_r^2+v_z^2=v_0^2-(r_0^2-r^2)\,\Omega^2+\frac{2q}{m}(\phi_0-\phi)-2\,\Omega\frac{q}{m}(r_0\,A_{\varphi_0}-r\,A_\varphi)-\\ -\frac{1}{r^2}\left[r_0\,v_{\varphi_0}+\frac{q}{m}(r_0\,A_{\varphi_0}-r\,A^\varphi)+(r_0^2-r^2)\,\Omega\right]^2. \qquad (15)$$

The second term in the right hand side of eq. (15) arises from the centrifugal force and the third term within the bracket from the Coriolis force. Assuming the thermal velocity to be small it is now seen that the line along which the bracket in eq. (15) has its minimum value is displaced from the field line $r\,A_\varphi=r_0\,A_{\varphi_0}$ by the influence of the Coriolis force. For the laws governing the motion of matter there exists an analogy between the influence of the Lorentz force and the Coriolis force (Chandrasekhar 1953, 1956; Lehnert 1954). In the present example the latter gives rise to an equivalent homogeneous magnetic field. This is also seen from eqs. (3)–(6) where the particle can be regarded to move in an equivalent magnetic field $\boldsymbol{B}^*=\boldsymbol{B}+2m\,\boldsymbol{\Omega}/q$. The Coriolis force has a tendency to break up the magnetic bottle allowing the particles to escape in the z-direction. However, if the magnetic field is strong enough the deviation of the particle path from the field lines will be small. In such a case eq. (15) shows that the particle will be stopped in its motion along a field line at least when $r_0>r$ and work has to be performed against the centrifugal field at the expense of the thermal energy.

Similar conclusions may be drawn from expression (13). The longitudinal electric field is still assumed to vanish which is likely to be a good approximation in many cases of interest. One example is given by Fig. 1 where the electrode surfaces are parallel with the magnetic field lines of the current loop and the latter become orthogonal to the electric field.[1]

From eqs. (13) and (15) is easily seen that the influence of the Coriolis force will be of minor importance and the particles move almost along the field lines if the largest occurring angular velocity $\Omega_{\max}$ is much less than the smallest occurring gyro-frequency $\omega_{\min}$. This is also what is expected from the form of $\boldsymbol{B}^*$. Further, the minimum modulus of the centrifugal term can be made to exceed v_0^2 in spite of $\Omega_{\max}/\omega_{\min}\ll 1$, provided that the magnetic field is strong enough. In such cases a particle will be prevented from passing the mirrors and the confinement becomes complete. This is also plausible from the point of view of eq. (14) when $\phi=\phi_0$ along a field line, $r_0\gg r$ and the velocity of rotation much greater than u_0. Then, u_{φ_0} will always differ from zero and the bracket in eq. (14) has a modulus greater than u_0^2 along the field lines through r_0.

In configurations such as that given by Fig. 1 large differences r_0-r can be obtained. Then, the work against the centrifugal force can be made to exceed the thermal energy for a large group of particles. A numerical example may serve as an illustration to this result. Using the current loop of Fig. 1 with a radius of 0.5 m and $r_0\approx 2$ m, $r\approx 0.35$ m a mirror ratio of about 20 is obtained. With a magnetic field strength $B_{\min}\approx 0.5$ Vs/m^2 at r_0 and an electric field $E_{\min}\approx 10^6$ V/m the corresponding angular velocity becomes $\Omega\approx 1.5\times 10^6\,s^{-1}$ with a velocity of rotation $\Omega r_0 \approx 3 \times 10^6$ m/s. Deuterons with a thermal velocity of about 3×10^6 m/s have a mean energy corresponding to 7×10^8 °K, a largest radius of gyration $a_{\max} \approx 0.06$ m and a smallest gyro frequency $\omega_{\min} \approx 2 \times 10^7\ s^{-1}$. Consider the region between the field lines passing through the plane of the loop at a distance of 0.8 and 1.2 m from the axis. At the bounding field lines the second term within the bracket of eq. (13) will then have a modulus which is at least five times larger than that of the other two terms within the same bracket. Further, for deuterons with thermal velocities somewhat less than 3×10^6 m/s the third term in the bracket of eq. (13) will have a modulus somewhat larger than that of the first three terms in the same equation. Consequently, these deuterons will be trapped effectively in a potential trough centered around the field line $r A_\varphi = r_0 A_{\varphi_0}$ and blocked at the ends at r by means of the centrifugal force.

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[1] It is easily seen that Ω cannot be a function of r only for a particle which moves in vacuum in the static and perpendicular poloidal fields $\boldsymbol{E}$ and $\boldsymbol{B}$. If Ω is assumed to have the form $\Omega\,(r)$ the conditions $\boldsymbol{E}\cdot\boldsymbol{B}=0$, div. $\boldsymbol{B}=0$, curl $\boldsymbol{B}=0$ and div $\boldsymbol{E}=0$ would namely result in $\Omega=\text{const.}/r^2$, if $B_z\neq 0$. Then, curl $\boldsymbol{E}=0$ can only be satisfied if $\Omega=0$.

4. Concluding remarks

It should be pointed out that "heating" of particles at the start of a homopolar machine gives nearly cycloidal orbits where the velocity of rotation around the axis is equal to the velocity of gyration of the particle. By this mechanism ions will be heated much more than electrons and will lose thermal energy to the latter by collisions. This improves the reflection conditions for the ions but impairs the same conditions for the electrons. Thus, electrons are not trapped as effectively as ions and a charge separation and an electric field will be set up at the mirrors. Finally, this effect is counteracted by bremsstrahlung which radiates away a considerable amount of the thermal energy of the electrons.

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